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Effect of satisfying stress boundary conditions in the axisymmetric vibration analysis of circular and annular plates

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Abstract

In the present study, effect of satisfying stress boundary conditions, in addition to displacement boundary conditions, in the axisymmetric vibration analysis of circular and annular plates is investigated. A new axisymmetric finite element, which is based on a combination of the conventional displacement-type variational principle and Reissner's principle, is proposed. With this formulation, stresses, like displacements, are primary variables, and both displacement and stress boundary conditions can be easily and exactly imposed. Axisymmetric vibration frequencies of some typical circular and annular plates are then obtained with the present approach and are compared with those by the displacement-type axisymmetric finite element. Based on the results of the present method, it is found that the conventional finite element, though not satisfying stress boundary conditions, can still obtain sufficiently accurate vibration frequencies of circular and annular plates.

Keywords: Axisymmetric vibration; Stress boundary condition; Finite element

1. Introduction

To analyze the axisymmetric vibration of circular and annular plates, the finite element method is a useful and practical tool, no matter what theories it is based on. However, all the finite elements used to analyze the vibration of circular or annular plates to this date are of displacement-type formulations, e.g., Pardoen (1973), Gladwell and Vijay (1975), Guruswamy and Yang (1979) and Liu and Chen (1995). Displacement-type finite element formulations have the merit that they are simple in formulating, their applications are easy, and obtaining the frequencies and mode shapes are straightforward. Especially when analyzing the axisymmetric vibration of circular and annular plates with the axisymmetric finite element (Liu and Chen, 1995), the formulation is based on 3-D elasticity without any approximation and assumption, and all kinds of the displacement boundary conditions can be imposed exactly. This is not possible for other approaches in some cases (e.g., with different simply supported boundary conditions).

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Therefore, the axisymmetric finite element becomes an attractive tool in analyzing the vibration of circular and annular plates due to its simplicity, versatility, and accuracy (Liu and Lee, 2000). However, a question then was raised [1] if the free stress boundary conditions of the circular plates are satisfied. Mostly, the answer is 'no' due to the nature or a shortcoming of displacement-type finite element formulation. But, how large will be the effect of satisfying the stress boundary conditions, in addition to the displacement boundary conditions, compared with satisfying only the displacement boundary conditions when analyzing the vibration of circular and annular plates? In other words, the problem is how well the conventional displacement-type axisymmetric finite element can be trusted in analyzing the vibration of circular and annular plates. The present investigation represents an attempt to answer this problem. First, a new axisymmetric finite element formulation is proposed, which combines the conventional displacement-type variational form and Reissner's principle, such that the stresses as well as displacements are included as primary variables. The stress boundary conditions can then be satisfied easily and exactly. Vibration frequencies of some typical circular and annular plates are then derived with the present approach, and are compared to those obtained by the displacement-type axisymmetric finite element formulation to show the effect of additionally satisfying stress boundary conditions. Two more examples are also analyzed, and the results of a 3-D Ritz method (So and Leissa, 1998), a 3-D series method (Hutchinson and El-Azhari, 1986), a 2-D Mindlin Theory (Irie et al., 1980), the displacement-type axisymmetric finite element method (Liu and Chen, 1995), and the present method are presented to demonstrate the differences among these approaches.

2. Formulation

The conventional displacement-type finite element formulation for axisymmetric vibration of circular and annular plates uses the following variational form:

$$0 = \int_t \int_{\mathrm{vol}} [(\sigma_r \delta\varepsilon_r + \sigma_z \delta\varepsilon_z + \sigma_\theta \delta\varepsilon_\theta + \tau_{rz}\delta\gamma_{rz}) - \rho(\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w})]\,\mathrm{d}v\,\mathrm{d}t \tag{1}$$

where the strains can be expressed in terms of displacements u and w as

$$\varepsilon_r = \frac{\partial u}{\partial r} \quad \varepsilon_z = \frac{\partial w}{\partial z} \quad \varepsilon_\theta = \frac{u}{r} \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \tag{2}$$

where u and w are the displacements in the radial and the thickness directions, respectively, and are functions of the radial coordinate r, thickness coordinate z, and time t. They are not functions of the circumferential coordinate θ due to axisymmetry. The circumferential displacement v and strains $\gamma_{r\theta}$ and $\gamma_{z\theta}$ are all zero, and the circumferentially vibrating modes are not included in the present study.

With the above formulation, the primary variables are u and w only, stresses are the secondary variables, and cannot be imposed as boundary conditions. To satisfy the free stress boundary conditions, the variational form (1) is augmented with Reissner's principle (Washizu, 1975),

$$0 = \delta\int_t \left[\int_{\mathrm{vol}} [(\sigma_r\varepsilon_r + \sigma_z\varepsilon_z + \sigma_\theta\varepsilon_\theta + \tau_{rz}\gamma_{rz}) - (s_{11}\sigma_r^2 + s_{22}\sigma_z^2 + s_{33}\sigma_\theta^2 + s_{66}\tau_{rz}^2 + 2s_{12}\sigma_r\sigma_z + 2s_{13}\sigma_r\sigma_\theta + 2s_{23}\sigma_z\sigma_\theta)/2 - \rho(\dot{u}^2 + \dot{w}^2)/2]\,\mathrm{d}v \right] \mathrm{d}t \tag{3}$$

[1] Question raised by A.W. Leissa at the International Symposium on Vibrations of Continuous Systems, Estes Park, Colorado, 1997.

where s_{ij} are the compliance coefficients of the material. By combining Eqs. (1) and (3), the stresses now become dependent variables and their conditions can be imposed on the boundary, just like the displacements.

The finite element procedures are then followed, and the combined variational form will lead to an elemental equation as follows,

$$\begin{bmatrix} [m] & [0] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{\ddot{U}\} \\ \{\ddot{\sigma}\} \end{Bmatrix} + \begin{bmatrix} [k] & [b] \\ [b^{\mathrm{T}}] & [d] \end{bmatrix} \begin{Bmatrix} \{U\} \\ \{\sigma\} \end{Bmatrix} = 0 \tag{4}$$

where $\{U\}^{\mathrm{T}} = [u_1 \;\; u_2 \;\; \cdots \;\; u_n; \;\; w_1 \;\; w_2 \;\; \cdots \;\; w_n]$ and $\{\sigma\}^{\mathrm{T}} = [\sigma_{r1} \;\; \sigma_{r2} \;\; \cdots \;\; \sigma_{rn}; \sigma_{z1} \;\; \sigma_{z2} \;\; \cdots \;\; \sigma_{zn}; \;\; \sigma_{\theta 1} \;\; \sigma_{\theta 2} \;\; \cdots \;\; \sigma_{\theta n}; \;\; \tau_{rz1} \;\; \tau_{rz2} \;\; \cdots \;\; \tau_{rzn}]$, n is the number of nodes in an element, $[k]$ is the elemental stiffness matrix as usual, and $[b]$ and $[d]$ are derived matrices with the inclusion of Reissner's variational form. $[b^{\mathrm{T}}]$ is the transpose of $[b]$. Both $[k]$ and $[d]$ are symmetric. Details of these matrices are shown in the appendix.

Assemblage of the above equations for all elements yields the system equation,

$$\begin{bmatrix} [M] & [0] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{\ddot{X}\} \\ \{\ddot{\Sigma}\} \end{Bmatrix} + \begin{bmatrix} [K] & [B] \\ [B^{\mathrm{T}}] & [D] \end{bmatrix} \begin{Bmatrix} \{X\} \\ \{\Sigma\} \end{Bmatrix} = 0 \tag{5}$$

and the corresponding eigenvalue equation is

$$\begin{bmatrix} [K] & [B] \\ [B^{\mathrm{T}}] & [D] \end{bmatrix} \begin{Bmatrix} \{X\} \\ \{\Sigma\} \end{Bmatrix} = \lambda \begin{bmatrix} [M] & [0] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \{X\} \\ \{\Sigma\} \end{Bmatrix} \tag{6}$$

where the eigenvalue λ is the square of the vibration frequency ω. It should be noted that the above equation cannot be solved directly due to the appearing of the zero matrices, and some matrix manipulations should be conducted as follows before we can obtain the eigenvalues.

The lower part of Eq. (6) shows that

$$[B^{\mathrm{T}}]\{X\} + [D]\{\Sigma\} = 0$$

To impose the free stress boundary conditions, the rows of $[B^{\mathrm{T}}]$ and $[D]$ corresponding to the specified nodal free stresses are first set to zero, so are the corresponding columns of $[D]$, and then the diagonal entries of these rows in $[D]$ are set to 1. We then end up with the following equation, where the specified free stress boundary conditions are guaranteed. Also, matrices $[B^{\mathrm{T}}]$ and $[D]$ have been changed to $[B^{\mathrm{T}^*}]$ and $[D^*]$, respectively.

$$[B^{\mathrm{T}^*}]\{X\} + [D^*]\{\Sigma\} = 0 \quad \text{or} \quad \{\Sigma\} = -[D^*]^{-1}[B^{\mathrm{T}^*}]\{X\}$$

The above stress expression is then substituted into the upper part of Eq. (6), and we obtain

$$\left[[K] - [B][D^*]^{-1}[B^{\mathrm{T}^*}]\right]\{X\} = \lambda[M]\{X\} \tag{7}$$

Following rules of matrix multiplication, it is found that the above equation is equivalent to

$$\left[[K] - [B^*][D^*]^{-1}[B^{\mathrm{T}^*}]\right]\{X\} = \lambda[M]\{X\} \tag{8}$$

with $[B^*]$ being the transpose of $[B^{\mathrm{T}^*}]$. This is the eigenvalue equation we use to calculate the vibration frequencies of circular and annular plates. It is also found that the modified stiffness matrix in Eq. (8) is still symmetric.

3. Examples and discussion

To see the validation and accuracy of the present formulation and the difference between it and the conventional displacement-type formulation, axisymmetric vibration of some typical circular and annular plates with various combinations of radius-to-thickness ratios a/h, inner-to-outer radius ratios b/a (annular plates), and boundary conditions is analyzed, and the results are presented in Tables 2–6 and compared to those of the displacement-type finite element formulation which has been shown to be able to obtain accurate frequencies up to date (Liu and Lee, 2000). Five types of boundary conditions are studied including one clamped and four simply supported conditions. Fig. 1 shows the specified displacements and stresses on the boundaries for the various types of boundary conditions. The vibration frequency ω is nondimensionalized according to $\bar{\omega} = \omega(\rho h a^4/D)^{1/2}$ where ρ is density of material and $D = Eh^3/12(1 - \nu^2)$, with E

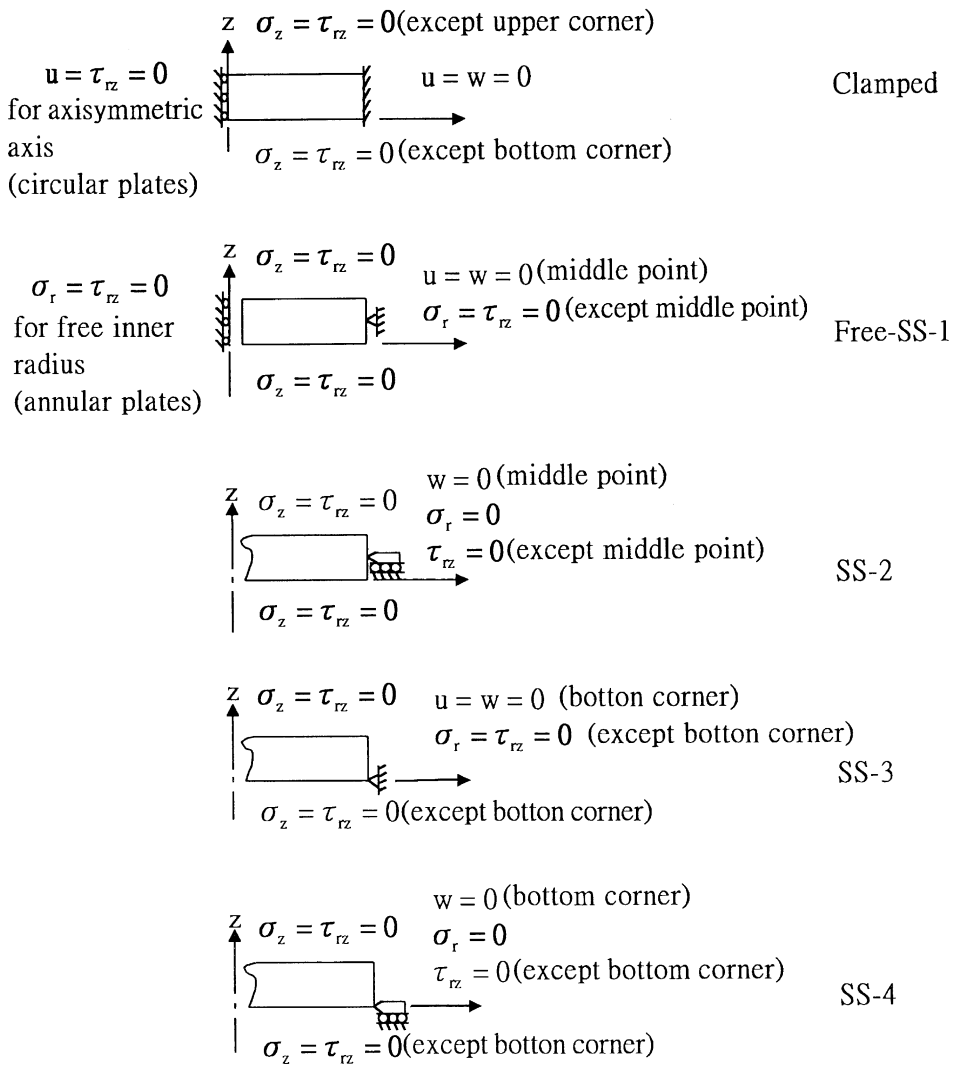

Fig. 1. Boundary conditions.

Table 1
Typical convergence of $\bar{\omega}_1$ of circular plates with mesh refinement

a/h	4	5	10	20	50
Clamped	(4×2) 8.991	(5×2) 9.358	(10×1) 10.059		
	(8×2) 8.935	(10×2) 9.350	(10×2) 10.016	(20×1) 10.200	(50×1) 10.225
	(8×4) 8.923	(10×4) 9.338	(20×2) 9.990	(20×2) 10.180	(50×2) 10.218
SS1 & SS2	(4×2) 4.705	(5×1) 4.796			
	(8×2) 4.697	(5×2) 4.784	(10×1) 4.899	(20×1) 4.926	(50×1) 4.934
	(8×4) 4.694	(10×2) 4.780	(10×2) 4.896	(20×2) 4.925	(50×2) 4.934
SS3	(4×2) 6.544	(5×1) 6.939			
	(8×2) 6.474	(5×2) 6.794	(10×1) 7.342	(20×1) 7.493	(50×1) 7.569
	(8×4) 6.367	(10×2) 6.729	(10×2) 7.232	(20×2) 7.451	(50×2) 7.539
SS4	(4×2) 4.621	(5×1) 4.743			
	(8×2) 4.596	(5×2) 4.736	(10×1) 4.890	(20×1) 4.924	(50×1) 4.934
	(8×4) 4.589	(10×2) 4.722	(10×2) 4.888	(20×2) 4.919	(50×2) 4.934

and v being Young's modulus and Poisson's ratio, respectively. The finite element employed is the 2-D eight-node isoparametric quadratic element in the rz-plane and the finite element meshes are successively refined by increasing the number of grids in the radial and the thickness directions alternatively. Typical convergence with mesh refinement is demonstrated in Table 1 wherein the numbers in the parentheses denote the number of grids in the radial and the thickness directions, respectively. Convergence is monotonic with the present formulation and is considered to be reached when the difference of frequencies between two consecutive meshes is less than 0.5% for both the mixed and displacement-type approaches. Convergence rates are also similar for both approaches, though different for different cases. Corresponding results from the two methods shown are of the same mesh. Almost all the cases in Tables 2–6 are convergent, except those with SS3 conditions and few others (with maximum 2.7% difference between the present mesh and the previous one). It should be noted that, although the examples analyzed here are the same as those in Liu and Chen (1995), the vibration frequencies obtained therein by the displacement-type formulation are different from those in the present results by the same formulation. This is due to different meshes used. Much refined meshes are employed in the present analysis.

From the results in Tables 2–6, we may find that vibration frequencies of both the mixed formulation and the conventional displacement approach (with a "+") are close to each other, except for those cases with SS3 conditions and the one with $a/h = 5$, $b/a = 0.1$, simply supported–simply supported in Table 5. In the latter case, only frequencies with the SS1–SS1 and SS2–SS2 boundary conditions and by mixed formulation satisfy the 0.5% convergence criterion. We may also find that (1) the nondimensional frequencies obtained by the present mixed formulation are always smaller than those by the conventional displacement-type axisymmetric finite element, (this may attribute to the "softening" effect of the complementary strain energy to the strain energy when Reissner's principle is incorporated.) and (2) the difference of vibration frequencies between these two methods becomes smaller in general as the thickness decreases. Numbers in parentheses in these five tables show the percentage differences of the fundamental frequencies from these two methods for each case, and a "$*$" behind a number represents that there exists a nonflexural mode just before the present one and its frequency is not shown.

To further justify the accuracy of the present mixed formulation, two additional examples of thick annular plates are analyzed. One has $a/h = 2.5$, $b/a = 0.5$, and the other $a/h = 1$, $b/a = 0.5$. Both are of free boundary. The nondimensionalization is $\bar{\omega} = \omega a\sqrt{(\rho/G)}$. In Table 7, the present results are compared with those by a 3-D Ritz method (So and Leissa, 1998), a 3-D series method (Hutchinson and El-Azhari, 1986), a 2-D Mindlin theory (Irie et al., 1980), and the 3-D displacement-type axisymmetric finite element

Table 2
Nondimensional frequencies for clamped circular plates

a/h	4	5	10	20	50	
Mode 1	8.923	9.338	9.990	10.180	10.218	Present
	8.930 (0.08%)	9.340 (0.02%)	9.997 (0.07%)	10.184 (0.04%)	10.219 (0.01%)	+
Mode 2	27.810	30.709	36.753	39.027	39.678	Present
	27.834	30.721	36.791	39.048	39.685	+
Mode 3	50.748	57.930	76.474	85.488	88.553	Present
	50.807	57.966	76.581	85.551	88.570	+
Mode 4	75.487*	87.900*	125.050	147.507	156.397	Present
	75.646*	87.998*	125.273	147.653	156.433	+

(Liu and Lee, 2000). It should be noted that a finite-term Ritz approach only satisfies the free stress boundary conditions approximately. Also, in Hutchinson and El-Ahari's series solution, part of the stress boundary conditions are satisfied only approximately. Therefore, all the above four methods do not fulfill the stress boundary conditions exactly, although they are considered to be among the most accurate known to date. (In Table 7, the frequencies given by So and Leissa (1998) are exact to the four or five significant figures, as discussed in their paper.) From Table 7, we may observe that the results of the present method, the displacement-type axisymmetric finite element, and the 3DR method agree very well with one another, with the present solutions lie between the other two.

4. Conclusion

In the present investigation, a mixed axisymmetric finite element formulation is proposed which has stresses, as well as displacements, as dependable variables. Therefore, in the analysis of axisymmetric vibration of circular and annular plates, the satisfying of the stress boundary conditions is accomplished just as easily and exactly as the displacement boundary conditions. With such a formulation, one of the most

Table 3
Nondimensional frequencies for simply supported circular plates

a/h	4	5	10	20	50	
Mode 1	4.694	4.780	4.896	4.925	4.934	SS1, Present
	4.694	4.780	4.896	4.925	4.934	SS2, Present
	6.367	6.729	7.232	7.451	7.539	SS3, Present
	4.589	4.722	4.888	4.919	4.934	SS4, Present
	4.699 (0.11%)	4.784 (0.08%)	4.897 (0.02%)	4.926 (0.02%)	4.934 (0.00%)	SS1, +
	4.699 (0.11%)	4.784 (0.08%)	4.897 (0.02%)	4.926 (0.02%)	4.934 (0.00%)	SS2, +
	6.509 (2.18%)	6.840 (1.62%)	7.317 (1.16%)	7.473 (0.29%)	7.559 (0.26%)	SS3, +
	4.598 (0.20%)	4.729 (0.15%)	4.889 (0.02%)	4.924 (0.10%)	4.933 (−0.02%)	SS4, +
Mode 2	23.092	24.985	28.276	29.345	29.659	SS1, Present
	23.092	24.985	28.276	29.345	29.659	SS2, Present
	22.285	25.351	30.516	32.026	32.692	SS3, Present
	21.138	23.823	28.139	29.192	29.657	SS4, Present
	23.203	25.074	28.322	29.348	29.659	SS1, +
	23.203	25.074	28.322	29.348	29.659	SS2, +
	22.617	25.669	30.669	32.208	32.726	SS3, +
	21.348	24.100	28.176	29.328	29.657	SS4, +

Table 4
Nondimensional frequencies for clamped–clamped annular plates

	a/h				
	5	10	20	50	
	$b/a = 0.1$				
Mode 1	20.256	24.919	26.689	27.243	Present
	20.278 (0.11%)	24.930 (0.04%)	26.702 (0.05%)	27.251 (0.03%)	+
Mode 2	46.027	63.062	71.756	74.915	Present
	46.078	63.094	71.788	74.940	+
$b/a = 0.3$					
Mode 1		39.980	43.925	45.238	Present
		40.000 (0.05%)	43.956 (0.07%)	45.255 (0.04%)	+
Mode 2		97.383	116.350	124.126	Present
		97.441	116.459	124.179	+
$b/a = 0.5$					
Mode 1		71.736	83.853	88.684	Present
		71.788 (0.07%)	83.880 (0.03%)	88.733 (0.06%)	+
Mode 2		163.892	214.592	241.275	Present
		164.041	214.671	241.431	+

severe disadvantages of the conventional displacement-type finite element formulation has been eliminated. From the results and comparisons shown, it may be found that vibration frequencies obtained by the present method are smaller than those by the conventional 3-D axisymmetric finite element method. However, the differences are so little that it may be concluded that the conventional formulation, though not satisfying the stress boundary conditions, can still obtain sufficiently accurate vibration frequencies of circular and annular plates, and should be considered first when performing an analysis, due to its simplicity and economy in computing.

Appendix A

$$[k] = \begin{bmatrix} [k_{11}] & [k_{12}] \\ [k_{21}] & [k_{22}] \end{bmatrix} \qquad [b] = \begin{bmatrix} [b_{11}] & [b_{12}] & [b_{13}] & [b_{14}] \\ [b_{21}] & [b_{22}] & [b_{23}] & [b_{24}] \end{bmatrix}$$

$$[m] = \begin{bmatrix} [m_{11}] & [m_{12}] \\ [m_{21}] & [m_{22}] \end{bmatrix} \qquad [d] = \begin{bmatrix} [d_{11}] & [d_{12}] & [d_{13}] & [d_{14}] \\ [d_{21}] & [d_{22}] & [d_{23}] & [d_{24}] \\ [d_{31}] & [d_{32}] & [d_{33}] & [d_{34}] \\ [d_{41}] & [d_{42}] & [d_{43}] & [d_{44}] \end{bmatrix}$$

where

$$(m_{11})_{ij} = (m_{22})_{ij} = 2\int \rho N_i N_j \,\mathrm{d}v$$

$$(m_{12})_{ij} = (m_{21})_{ij} = 0$$

$$(k_{11})_{ij} = \int (c_{11}N_{i,r}N_{j,r} + c_{13}N_{i,r}N_j/r + c_{13}NiN_{j,r}/r + c_{33}N_iN_j/r^2 + c_{66}N_{i,z}N_{j,z})\,\mathrm{d}v$$

Table 5
Nondimensional frequencies simply supported–simply supported annular plates

	a/h				
	5	10	20	50	
b/a = 0.1					
Mode 1	11.979	13.786	14.317	14.458	SS1–SS1, Present
	11.979	13.786	14.317	14.458	SS2–SS2, Present
	13.776	17.752	19.471	20.095	SS3–SS3, Present
	10.785	13.572	14.316	14.461	SS4–SS4, Present
	12.092 (0.93%)	13.808 (0.16%)	14.320 (0.02%)	14.458 (0.00%)	SS1–SS1, +
	12.092 (0.93%)	13.808 (0.16%)	14.320 (0.02%)	14.458 (0.00%)	SS2–SS2, +
	14.388 (4.25%)	18.126 (2.06%)	19.727 (1.30%)	20.213 (0.58%)	SS3–SS3, +
	10.938 (1.40%)	13.609 (0.27%)	14.323 (0.05%)	14.462 (0.01%)	SS4–SS4, +
Mode 2	36.396	46.520	50.410	51.556	SS1–SS1, Present
	36.396*	46.520	50.410	51.556	SS2–SS2, Present
	31.258	48.276	55.854	58.196	SS3–SS3, Present
	34.194*	44.738	50.264	51.561	SS4–SS4, Present
	36.934	46.660	50.435	51.560	SS1–SS1, +
	36.934*	46.660	50.435	51.560	SS2–SS2, +
	31.878	48.851	56.223	58.355	SS3–SS3, +
	34.200*	44.970	50.322	51.562	SS4–SS4, +
b/a = 0.3					
Mode 1		20.142	20.854	21.043	SS1–SS1, Present
		20.142	20.854	21.043	SS2–SS2, Present
		27.444	30.629	31.881	SS3–SS3, Present
		19.787	20.826	21.043	SS4–SS4, Present
		20.168 (0.13%)	20.858 (0.02%)	21.044 (0.00%)	SS1–SS1, +
		20.168 (0.13%)	20.858 (0.02%)	21.044 (0.00%)	SS2–SS2, +
		28.124 (2.42%)	31.107 (1.54%)	32.063 (0.57%)	SS3–SS3, +
		19.831 (0.22%)	20.836 (0.05%)	21.044 (0.00%)	SS4–SS4, +
Mode 2		71.096	78.841	81.259	SS1–SS1, Present
		71.096*	78.841	81.259	SS2–SS2, Present
		64.396	81.523	86.780	SS3–SS3, Present
		67.258*	78.384	81.240	SS4–SS4, Present
		71.353	78.893	81.261	SS1–SS1, +
		71.353*	78.893	81.261	SS2–SS2, +
		64.714	81.710	86.845	SS3–SS3, +
		67.693*	78.508	81.238	SS4–SS4, +
b/a = 0.5					
Mode 1		37.045	39.307	39.925	SS1–SS1, Present
		37.045	39.307	39.925	SS2–SS2, Present
		49.146	57.874	61.171	SS3–SS3, Present
		35.574	39.149	39.916	SS4–SS4, Present
		37.144 (0.27%)	39.324 (0.04%)	39.927 (0.01%)	SS1–SS1, +
		37.144 (0.27%)	39.324 (0.04%)	39.927 (0.01%)	SS2–SS2, +
		50.756 (3.17%)	59.047 (1.99%)	61.716 (0.88%)	SS3–SS3, +
		35.735 (0.45%)	39.190 (0.10%)	39.919 (0.01%)	SS4–SS4, +
Mode 2		124.626	148.448	156.878	SS1–SS1, Present
		124.626*	148.448*	156.878	SS2–SS2, Present
		85.794	138.478	159.302	SS3–SS3, Present
		110.419*	146.264*	156.715	SS4–SS4, Present
		125.585	148.677	156.899	SS1–SS1, +
		125.585*	148.677*	156.899	SS2–SS2, +
		87.402	138.983	159.358	SS3–SS3, +
		111.909*	146.806*	156.761	SS4–SS4, +

$$(k_{12})_{ij} = \int (c_{12}N_{i,r}N_{j,z} + c_{23}NiN_{j,z}/r + c_{33}N_iN_j/r^2 + c_{66}N_{i,z}N_{j,r})\,\mathrm{d}v$$

$$(k_{12})_{ij} = (k_{12})_{ji}$$

$$(k_{22})_{ij} = \int (c_{22}N_{i,z}N_{j,z} + c_{66}N_{i,r}N_{j,r})\,\mathrm{d}v$$

$i, j = 1 \sim n$, and c_{pq} are the stiffness coefficients.

$$(b_{11})_{ij} = \int N_{i,r}N_j\,\mathrm{d}v$$

$$(b_{13})_{ij} = \int (N_iN_j/r)\,\mathrm{d}v$$

$$(b_{14})_{ij} = \int N_{i,z}N_j\,\mathrm{d}v$$

$$(b_{22})_{ij} = \int N_{i,z}N_j\,\mathrm{d}v$$

$$(b_{24})_{ij} = \int N_{i,r}N_j\,\mathrm{d}v$$

$$(d_{11})_{ij} = \int -s_{11}N_iN_j\,\mathrm{d}v$$

$$(d_{12})_{ij} = \int -s_{12}N_iN_j\,\mathrm{d}v$$

$$(d_{13})_{ij} = \int -s_{13}N_iN_j\,\mathrm{d}v$$

$$(d_{21})_{ij} = (d_{12})_{ji}$$

$$(d_{22})_{ij} = \int -s_{22}N_iN_j\,\mathrm{d}v$$

$$(d_{23})_{ij} = \int -s_{23}N_iN_j\,\mathrm{d}v$$

$$(d_{31})_{ij} = (d_{13})_{ji}$$

Table 6
Nondimensional frequencies for free–simply supported annular plates

	a/h				
	5	10	20	50	
b/a = 0.1					
Mode 1	4.710	4.817	4.844	4.852	Free–SS1, Present
	4.710	4.817	4.844	4.852	Free–SS2, Present
	6.716	7.151	7.339	7.475	Free–SS3, Present
	4.663	4.809	4.843	4.852	Free–SS4, Present
	4.716 (0.13%)	4.818 (0.02%)	4.844 (0.00%)	4.852 (0.00%)	Free–SS1, +
	4.716 (0.13%)	4.818 (0.02%)	4.844 (0.00%)	4.852 (0.00%)	Free–SS2, +
	6.856 (2.04%)	7.235 (1.16%)	7.386 (0.64%)	7.469 (−0.08%)	Free–SS3, +
	4.672 (0.19%)	4.811 (0.04%)	4.843 (0.00%)	4.852 (0.00%)	Free–SS4, +
Mode 2	25.012	28.096	29.080	29.381	Free–SS1, Present
	25.012	28.096	29.080	29.381	Free–SS2, Present
	25.451	30.259	31.818	32.395	Free–SS3, Present
	24.051	27.928	29.057	29.380	Free–SS4, Present
	25.118	28.117	29.087	29.381	Free–SS1, +
	25.118	28.117	29.087	29.381	Free–SS2, +
	25.776	30.412	31.895	32.393	Free–SS3, +
	24.263	27.971	29.064	29.379	Free–SS4, +
b/a = 0.3					
Mode 1		4.632	4.656	4.663	Free–SS1, Present
		4.632	4.656	4.663	Free–SS2, Present
		7.279	7.470	7.594	Free–SS3, Present
		4.624	4.655	4.663	Free–SS4, Present
		4.632 (0.00%)	4.656 (0.00%)	4.663 (0.00%)	Free–SS1, +
		4.632 (0.00%)	4.656 (0.00%)	4.663 (0.00%)	Free–SS2, +
		7.363 (1.14%)	7.516 (0.61%)	7.600 (0.08%)	Free–SS3, +
		4.625 (0.02%)	4.655 (0.00%)	4.663 (0.00%)	Free–SS4, +
Mode 2		34.977	36.489	36.953	Free–SS1, Present
		34.977	36.489	36.953	Free–SS2, Present
		36.706	39.362	40.235	Free–SS3, Present
		34.639	36.437	36.950	Free–SS4, Present
		35.009	36.494	36.953	Free–SS1, +
		35.009	36.494	36.953	Free–SS2, +
		36.854	39.437	40.256	Free–SS3, +
		34.715	36.448	36.949	Free–SS4, +
b/a = 0.5					
Mode 1		5.034	5.066	5.075	Free–SS1, Present
		5.034	5.066	5.075	Free–SS2, Present
		8.663	8.912	9.043	Free–SS3, Present
		5.025	5.064	5.075	Free–SS4, Present
		5.035 (0.02%)	5.066 (0.00%)	5.075 (0.00%)	Free–SS1, +
		5.035 (0.02%)	5.066 (0.00%)	5.075 (0.00%)	Free–SS2, +
		8.751 (1.01%)	8.961 (0.55%)	9.063 (0.22%)	Free–SS3, +
		5.026 (0.02%)	5.064 (0.00%)	5.075 (0.00%)	Free–SS4, +
Mode 2		59.618	64.089	65.553	Free–SS1, Present
		59.618*	64.089	65.553	Free–SS2, Present
		55.894	65.761	68.762	Free–SS3, Present
		58.238*	63.866	65.535	Free–SS4, Present
		59.751	64.111	65.555	Free–SS1, +
		59.751*	64.111	65.555	Free–SS2, +
		56.147	65.831	68.787	Free–SS3, +
		58.557*	63.918	65.539	Free–SS4, +

Table 7
Comparison of nondimensional axisymmetric, flexural vibration frequencies of the annular thick plates among the present, 3-D displacement type, axisymmetric finite element (3DD), 3-D Ritz (3DR), 3-D Hutchinson's series method (3DH) and 2-D Mindlin theory (2DM), with $a/h = 2.5$, $b/a = 0.5$ (mesh 10×4) and $a/h = 1$, $b/a = 0.5$ (mesh 8×8)

a/h	b/a	Method	Mode			
			1	2	3	4
2.5	0.5	Present	1.388	8.323	9.131	14.150
		3DD	1.388	8.324	9.132	14.160
		3DR	1.388	8.321	9.127	14.133
		3DH	1.398	8.327	9.128	10.398
		2DM	1.388	8.324	9.370	10.593
1	0.5	Present	1.984	5.774	8.265	9.086
		3DD	1.984	5.774	8.268	9.087
		3DR	1.984	5.772	8.258	9.084
		3DH	1.985	5.774	7.503	8.259
		2DM	1.985	6.720	7.547	10.010

$$(d_{32})_{ij} = (d_{23})_{ji}$$

$$(d_{33})_{ij} = \int -s_{33} N_i N_j \, \mathrm{d}v$$

$$(d_{44})_{ij} = \int -s_{66} N_i N_j \, \mathrm{d}v$$

All other matrices of $(b_{pq})_{ij}$ and $(d_{pq})_{ij}$ are zero.

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